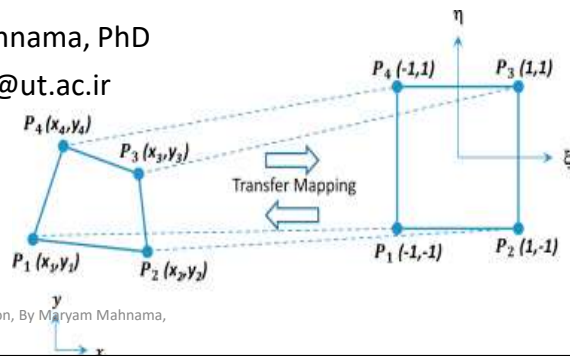




Chapter 9: Isoparametric Formulation

Higher-order Elements

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Higher-Order Shape Functions



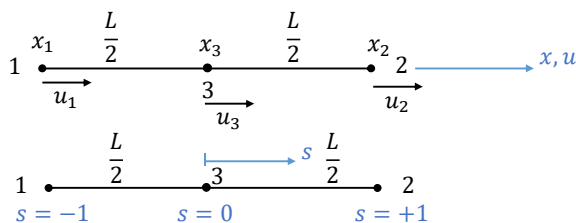
- Higher-order element shape functions can be developed by adding additional nodes to the sides of the linear element.



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Three-noded Linear Strain Bar Isoparametric Element

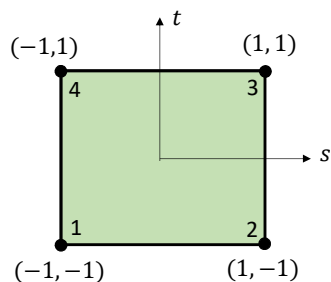


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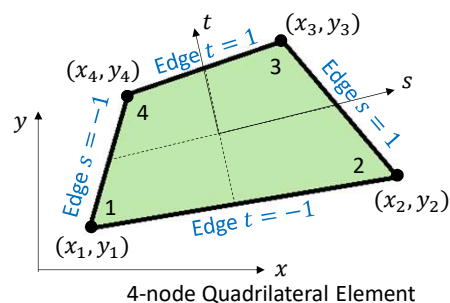
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Isoparametric Formulation of the Quadrilateral Element Stiffness Matrix



Transformation
Mapping



4-node Quadrilateral Element

A rectangular element with $h = b = 1$
The shape functions as well as stiffness
matrix are known for this element.

We should obtain some expressions to
map (x, y) into (s, t) coordinate system.

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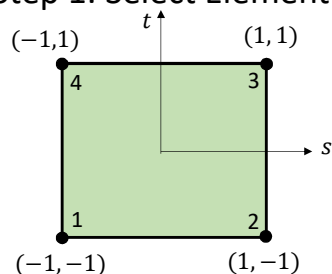
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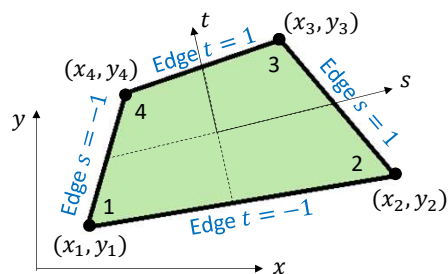
Isoparametric Formulation of the Quadrilateral Element Stiffness Matrix (Step1)



- Step 1: Select Element Type



Transformation Mapping



$$x = a_1 + a_2 s + a_3 t + a_4 st$$

$$y = a_5 + a_6 s + a_7 t + a_8 st$$

4 BCs in both directions

$$\begin{aligned} (-1, -1) &\Leftarrow (x_1, y_1) \\ (1, -1) &\Leftarrow (x_2, y_2) \\ (1, 1) &\Leftarrow (x_3, y_3) \\ (-1, 1) &\Leftarrow (x_4, y_4) \end{aligned}$$

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Gaussian Quadrature



- How can one evaluate the integral

$$I = \int_{-1}^1 y dx \cong 2y(0)$$

one-point Gaussian quadrature

It is only needed to evaluate the function at one point

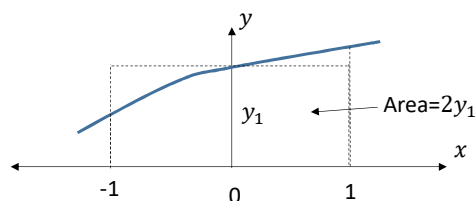
- This formula can be generalize as

$$I = \int_{-1}^1 y dx = \sum_{i=1}^n W_i y_i$$

Weighting

Value of function at sampled point

It is needed to evaluate the function at n points



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Gaussian Quadrature



- The rules of Gaussian Quadrature:

Gauss's method chooses the sampling points so that for a given number of points, the best possible accuracy is obtained.

1. Sampling points are located symmetrically with respect to the center of the interval.

2. Symmetrically paired points are given the same weight W_i

Number of Points	Locations, x_i	Associated Weights, W_i
1	$x_1 = 0.000 \dots$	2.000
2	$x_1, x_2 = \pm 0.57735026918962$	1.000
3	$x_1, x_3 = \pm 0.77459666924148$ $x_2 = 0.000 \dots$	$\frac{5}{9} = 0.555 \dots$ $\frac{8}{9} = 0.888 \dots$
4	$x_1, x_4 = \pm 0.8611363116$ $x_2, x_3 = \pm 0.3399810436$	0.3478548451 0.6521451549

v

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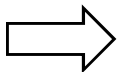


Gaussian Quadrature



- Note: Gaussian quadrature using n points (Gauss points) is exact if the integrand is a polynomial of degree $2n - 1$ or less.

In using n points, we effectively replace the given function $y = f(x)$ by a polynomial of degree $2n - 1$. The accuracy of the numerical integration depends on how well the polynomial fits the given curve.



If the function $f(x)$ is **not a polynomial**, Gaussian quadrature is inexact, but it becomes more accurate as more Gauss points are used.

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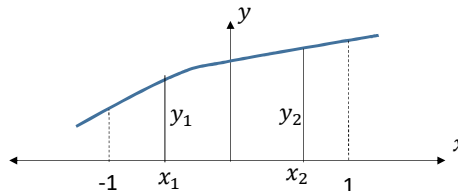
Two-Point Gaussian Quadrature



- We want to know how the x_i s and W_i s are obtained in Gauss's method:

$$I = \int_{-1}^1 y dx = W_1 y_1 + W_2 y_2 = \boxed{W_1} y(\boxed{x_1}) + \boxed{W_2} y(\boxed{x_2})$$

There are four unknowns



⇒ We assume a cubic function for y as $y = C_0 + C_1 x + C_2 x^2 + C_3 x^3$

With four parameters in the two-point formula, we would expect the Gauss formula to exactly predict the area under the curve.

$$A = \int_{-1}^1 y dx = \int_{-1}^1 C_0 + C_1 x + C_2 x^2 + C_3 x^3 dx = W_1 y(x_1) + W_2 y(x_2)$$



Two-Point Gaussian Quadrature



$$A = \int_{-1}^1 y dx = \int_{-1}^1 C_0 + C_1 x + C_2 x^2 + C_3 x^3 dx = 2C_0 + \frac{2}{3}C_2$$

- According to Gauss's rules $W_1 = W_2 = W$ and $x_1 = -x_2 = a$
- The area predicted by Gauss's formula is

$$A_G = W y(a) + W y(-a) = 2WC_0 + 2a^2WC_2$$

- If the error, $e = A - A_G$, is to vanish for any C_0 and C_2 , we must have

$$e = 2C_0(1 - W) + 2C_2\left(\frac{1}{3} - a^2W\right)$$

$$\frac{\partial e}{\partial C_0} = 0 \Rightarrow 2(1 - W) = 0 \Rightarrow W = 1$$

$$x_1 = -x_2 = 1/\sqrt{3} = 0.5773$$

$$W_1 = W_2 = 1$$

$$\frac{\partial e}{\partial C_2} = 0 \Rightarrow 2\left(\frac{1}{3} - a^2W\right) = 0 \Rightarrow a = \frac{1}{\sqrt{3}}$$



Example



- We want to find the integral $I = \int_{-1}^1 [x^2 + \cos(x/2)] dx$ by using 3-point Gaussian quadrature.

Using the table, we have $x_1 = x_3 = \pm 0.77459$ and $x_2 = 0$. Also $W_1 = W_3 = \frac{5}{9}$ and $W_2 = \frac{8}{9}$. So,

$$I = \left[(-0.77459)^2 + \cos\left(-\frac{0.77459}{2} \text{ rad}\right) \right] \frac{5}{9} + \left[(0)^2 + \cos\left(\frac{0}{2} \text{ rad}\right) \right] \frac{8}{9} + \left[(0.77459)^2 + \cos\left(\frac{0.77459}{2} \text{ rad}\right) \right] \frac{5}{9} = 2.585$$

$$I_{\text{exact}} = 2.585$$



Gaussian Quadrature in 2D



- In two dimensions, we obtain the quadrature formula by integrating first with respect to one coordinate and then with respect to the other as

$$I = \int_{-1}^1 \int_{-1}^1 f(s, t) ds dt = \int_{-1}^1 \left[\sum_i W_i f(s_i, t) \right] dt = \sum_j W_j \left[\sum_i W_i f(s_i, t_j) \right] = \sum_j \sum_i W_i W_j f(s_i, t_j)$$

- In the case of 4 Gauss points (2×2 Gauss points)

$$I = W_1 W_1 f(s_1, t_1) + W_1 W_2 f(s_1, t_2) + W_2 W_1 f(s_2, t_1) + W_2 W_2 f(s_2, t_2)$$



Evaluation of the Stiffness Matrix and Stress Matrix by Gaussian Quadrature

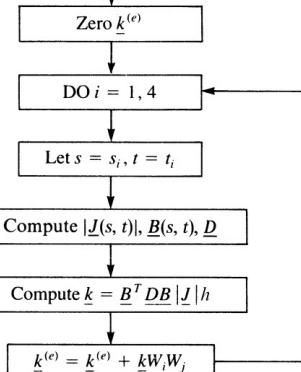


• Evaluation of the Stiffness Matrix

$$[k] = \int_{-1}^1 \int_{-1}^1 [B(s, t)]^T [D] [B(s, t)] h |J(s, t)| ds dt$$

$$\begin{aligned} \underline{k} = & \underline{B}^T(s_1, t_1) \underline{D} \underline{B}(s_1, t_1) |\underline{J}(s_1, t_1)| h W_1 W_1 \\ & + \underline{B}^T(s_2, t_2) \underline{D} \underline{B}(s_2, t_2) |\underline{J}(s_2, t_2)| h W_2 W_2 \\ & + \underline{B}^T(s_3, t_3) \underline{D} \underline{B}(s_3, t_3) |\underline{J}(s_3, t_3)| h W_3 W_3 \\ & + \underline{B}^T(s_4, t_4) \underline{D} \underline{B}(s_4, t_4) |\underline{J}(s_4, t_4)| h W_4 W_4 \end{aligned}$$

Read in four Gauss points and weight functions
 $s_i, t_i = \pm 0.5773 \dots; W_i, W_j = 1, 1.$



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Evaluation of the Stiffness Matrix and Stress Matrix by Gaussian Quadrature



• Evaluation of Element Stresses

There are three ways to compute stress in the element:

$$\{\sigma\} = [D][B(s, t)]\{d\}$$

A function of s and t

1. In practice, the stresses are evaluated at the same Gauss points used to evaluate the stiffness matrix



For a quadrilateral using 2×2 integration, we get four sets of stress data

2. It is also practical to compute the stresses at $s = t = 0$ in the mid-point of the element

3. To evaluate the stresses in all elements at a shared (common) node and then use an average of these element nodal stresses to represent the stress at the node.

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Summary



- Gaussian quadrature is the conventional method of numerical integration in FEM.
- Employing this technique, the integration is replaced by summation of the value of integrand at some points times some weightings.
- Employing n Gauss points, the integral can be computed precisely if the integrand is of order $2n-1$.
- Stiffness matrix, force vectors and stress values can be computed using Gauss points.